Individual Peeling of Multiple Singular Points

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It is shown that certain systems may exhibit multiple generating singular points which individually peel to result exploded points or limit cycles.

It was recently shown that the concept of exploded points, i.e. the objects of zero f-dimension, leads to detecting possible objects resulting from bifurcation of a generating singular point [1]. An illustrative example given was the Lorenz equation. The characteristics of an exploded point are that the observed object can not be a periodic orbit such as a limit cycle which is of f-dimension one.

An exploded point may not, however be the result of a bifurcation. Its existence may be possible in nonbifurcating systems. An exploded point as any other singular point may be either stable or unstable, or intuitively, attracting or repelling the trajectories in the surrounding neighborhood. If X ($\dim X = n$) is the space of interest, an exploded point $x_e \in X$ and taking x_e as the zero f-dimensional subspace of X, $x_e \subset X$, $\dim x_e = 0$, $\dim X = n$.

If a system goes through some bifurcations and results in some "chaotic objects", by the technique of detecting an exploded point, the dimension and the nature of the chaotic object may be determined. The bifurcation analysis of the Lorenz system, (1), given as an example for exploded points in [1] is discussed in detail in [2]. It is also noted in [1] that the Lorenz system cooperatively peels [4] which is an important concept that will be used in analyzing the objects appearing as a result of bifurcations.

In this paper we refer to two more examples (2) and (3) taken from [5].

(1) Lorenz

$$\dot{x} = -mx + my, \quad \dot{y} = -x(z-r) - y, \quad \dot{z} = xy - bz,$$

(3) Lorenz-like

$$\dot{x} = x - xy - z, \quad \dot{y} = x^2 - ay, \quad \dot{z} = bx - cz + d,$$

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(2) Screw-Type

$$\dot{x} = -y - z, \quad \dot{y} = x + ay, \quad \dot{z} = b + xz - cz.$$

The bifurcation analysis of (2) is given in [6]. Referring to Table 1 in [6] we can see that for $a < \sqrt{2}$, and $c > 2a$ the decomposed peeling of the stable generating singular point yields eigenvalues for the two bifurcated points such that

$$1\,w^1_{m_2} \oplus 1\,w^1_{m_1} \oplus 2\,w^1_{m_2},$$

yielding collectively $2w^1_{m_2}$. Thus the object for the screw-type chaos has the dimension $2-1=1$, a stable limit cycle is expected. In fact in [6] it is detected that a limit bundle exists which is basically a limit cycle with a very long period, see Fig. 2 in [6].

We will now discuss the third system, (3) and conclude some interesting results exhibited by this system.

The equations are taken as (3) i, ii, and iii. Here the parameters are a, b, c, and d. The singular solutions are found by setting $\dot{x} = \dot{y} = \dot{z} = 0$ to yield

$$y = x^2/a, \tag{4}$$

$$z = (bx + d)/c, \tag{5}$$

$$x^3 + a\left(\frac{b}{c} - 1\right)x + \frac{ad}{c} = 0. \tag{6}$$

Denoting

$$A = a\left(\frac{b}{c} - 1\right), \tag{7}$$

$$B = ad/c, \tag{8}$$

and noting that since the coefficient of the x^2 term in (6) is zero

$$x_1 + x_2 + x_3 = 0, \tag{9}$$

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where x's are either real or imaginary, the following cases for the solutions of Eq. (6) are found, see Table 1, and Figure 1.

The change in the number of singular solutions takes place when B crosses $B_2{}^*$ and $B_1{}^*$. To find the values of $B_i{}^*$ we notice that in Fig. 1 the slope of (6) must vanish at P_1 and P_2, thus differentiating (6) with respect to x we obtain

$$x = \pm\left[\frac{1}{3}a\left(1-\frac{b}{c}\right)\right]^{1/2} \tag{10}$$

or in terms of A,

$$x = \pm[-A/3]^{1/2}. \tag{10'}$$

This shows that this case is possible only in the case of $A<0$ as it can be seen in Figure 1b. Substituting the x-values from (10) in (6) one obtains,

$$\text{At } P_1;\quad x_{p1} = -(-A/3)^{1/2},\qquad B_1{}^* = \tfrac{2}{3}A(-A/3)^{1/2}, \tag{11}$$

$$\text{At } P_2;\quad x_{p2} = +(-A/3)^{1/2},\qquad B_2{}^* = -\tfrac{2}{3}A(-A/3)^{1/2}. \tag{12}$$

From (11), by substituting (7) for A and (8) for B's

$$\text{At } P_1:\quad d_1{}^* = \tfrac{2}{3}(b-c)[-a(b/c-1)/3]^{1/2}, \tag{13}$$

$$\text{At } P_2:\quad d_2{}^* = -\tfrac{2}{3}(b-c)[-a(b/c-1)/3]^{1/2}. \tag{14}$$

Referring to Table 1,

$d < d_2{}^*$	one singular point,
$d = d_2{}^*$	two singular points, one being a double point,
$d_2{}^* < d < d_1{}^*$	three singular points,
$d = d_1{}^*$	two singular points, one being a double point,
$d > d_1{}^*$	one singular point.

Using Equation (9),

At P_1: $x_1 = x_2 < 0$ and $x_3 > 0$, $-2x_1 = x_3$,

At P_2: $x_1 < 0$ and $x_2 = x_3 > 0$, $x_1 = -2x_3$.

Dividing (6) by $(x - x_3)$, we obtain

$$x^2 + x_3 x + (A + x_3{}^2) = 0. \tag{15}$$

Solving this quadratic equation, it is found that

$$x_{1,2} = [-x_3 \mp (-4A - 3x_3{}^2)^{1/2}]/2.$$

Evaluating this result at P_1 and P_2 one finds,

At P_1: $(x_1 = x_2)$, $x_{1,2} = \mp(-A/3)^{1/2}$,

$x_3 = 2(-A/3)^{1/2}$,

Table 1. Real solutions of Equation (6) for various A and B values.

A \ B	$B_2{}^* > 0$: $> B_2{}^*$	$= B_2{}^*$	$B_2{}^* >$	0	$B_1{}^* < 0$: $> B_1{}^*$	$= B_1{}^*$	$B_1{}^* >$
< 0	$x_1 < 0$	$x_1 < 0$, $x_2 = x_3 > 0$	$x_1 < 0$, $x_2 > 0$, $x_3 > 0$	$x_1 < 0$, $x_2 = 0$, $x_3 > 0$	$x_1 < 0$, $x_2 < 0$, $x_3 > 0$	$x_1 = x_2 < 0$, $x_3 > 0$	$x_3 > 0$
$= 0$		$x_1 < 0$		$x_1 = 0$		$x_3 > 0$	
> 0		$x_1 < 0$		$x_1 = 0$		$x_3 > 0$	

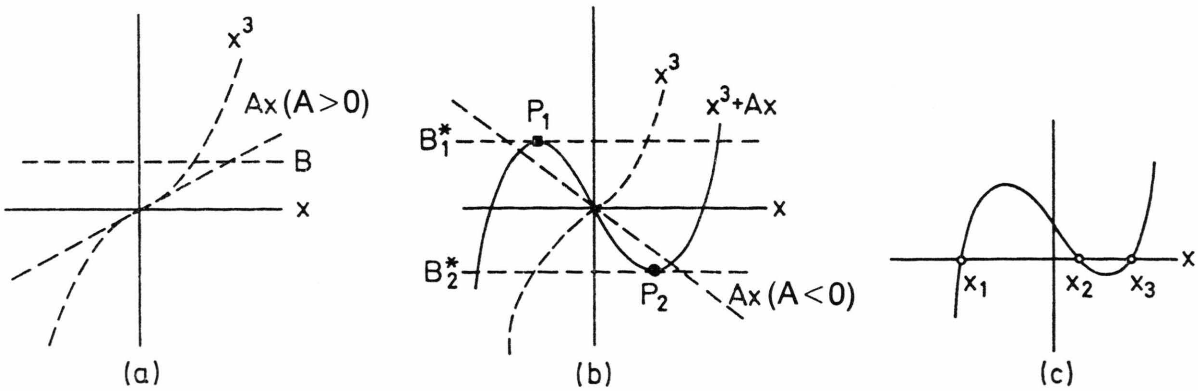

Fig. 1. Singular solutions of the System (3).

At P_2: $(x_2 = x_3)$, $x_1 = -2(-A/3)^{1/2}$,
$x_{2,3} = \mp(-A/3)^{1/2}$.

The Hessian for the systen (3) can easily be formed as

$$\Delta = \begin{vmatrix} (1-y) & -x & -1 \\ 2x & -a & 0 \\ b & 0 & -c \end{vmatrix}$$
$$= (1-y)ac - 2cx^2 - ab\,. \qquad (16)$$

Evaluating (16) at a solution where $y = x^2/a$ (Eq. (4)),

$$\Delta = (1-3y)ac - ab$$
$$= a[(1-3y)c - b]\,. \qquad (16')$$

From Table 1 it can be seen that a change in the number of singular solutions take place for either $A = 0$, or $B = B_1^*$ or B_2^*. These two cases yield the following results.

Case I:

$A = 0$ is possible if either $a = 0$, or $b = c$ (see (7)).
$a = 0$ yields $\Delta = 0$.
$b = c$ yields $\Delta = -3ayb$ or $-3x^2b$
where b can not vanish, see (6),
y vanishes implies $x = 0$, in turn either a or $d = 0$.

Thus $d = 0$ yields $\Delta = 0$. (Notice that $B = 0$.)

Case II:

$B = B_1^*$ or B_2^* is possible, see (11) and (12).
$a = 0$ is the same result as above,
$(1-3y^*)c - b = 0$ must be evaluated at the singular points.

Referring to (4), and the solutions obtained from (15).
At $x_{1,2}$ $(1 + A/a)c - b = 0$ is satisfied for all a, b, c.
At x_3 the result is $b = c$.
For P_2 the same values are found for the double point x_2, x_3 and the single point x_1 corresponding to the x_1, x_2 and x_3 above, respectively.

The stability consideration at singular points is based on the linearized equations about these points where the characteristic equations become,

$$\begin{vmatrix} (1-y^*)-s & -x^* & -1 \\ 2x^* & -a-s & 0 \\ b & 0 & -c-s \end{vmatrix} = 0$$

which yields

$$-s^3 + (1-a-c-y)s^2 + [a(1-c) + (c-b) - y(c+3a)]s + a[(c-b) - 3cy] = 0\,. \qquad (17)$$

Based on the analysis of the number of singular solutions, Table 1, and the parameter values where the Hessian vanishes we can arrive at the conclusion that

$$a = 0\,, \quad b/c = 1\,, \quad d = 0 \qquad (18)$$

are candidates to be considered in the bifurcation analysis. Studying the stability properties of the generating solutions for various combinations of these parameters we see that the system has one stable and one unstable part as in the case of the Lorenz system, [3]. These cases are summarized below:

Stable system:

$a > 0$, $b/c < 1$, and $d > 0$,
$a < 0$, $b/c > 1$, and $d < 0$;

Unstable system:

$a > 0$, $b/c < 1$, and $d < 0$,
$a < 0$, $b/c > 1$, and $d > 0$.

These regions are shown in Figure 2.

Discussion and Conclusions

As it was done for the systems (1) and (2), the third system can also be completely studied via the bifurcation analysis qualitatively. The following conclusions are significant.

	Change in Number of Sing. Sol.	$\Delta = 0$	Stability Changes	Bifurcation Parameter
$a = 0$	Yes	Yes	Yes	Yes
$b/c = 1$	Yes	Yes	Yes	Yes
$d = 0$	Yes	Yes	No	No

Here d basically determines the global stability of the system, Figure 2.

Conclusion 1: If a and b/c values are fixed, for $d > 0$ (or < 0) the system is globally stable (or unstable), d is not a bifurcation parameter.

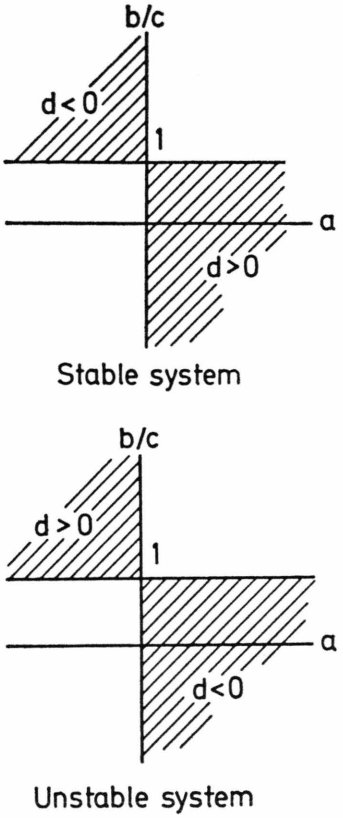

Fig. 2. Stable and unstable system regions corresponding to the parameter values.

The generating singular point, x_1, for the stable system has eigenvalues as follows:

$$s_1 < 0, \quad \operatorname{Re} s_2 < 0, \quad \operatorname{Re} s_3 < 0 \qquad \text{(Stable focus)}.$$

On the other hand, that of the unstable system possesses,

$$s_1 > 0, \quad \operatorname{Re} s_2 < 0, \quad \operatorname{Re} s_3 < 0. \qquad \text{(Saddle-focus)}.$$

We are interested in the stable system. There are two successive bifurcation phenomena, see Table 1. The first one is the creation of a *second generating* point:

$$x_1 \to (x_1, x_{2,3})$$

which takes place at $B_2{}^*$ value. The characteristic values are:

$$\text{Before:} \quad x_1; \quad 3\,w_{m_1},$$
$$\text{After:} \quad x_1; \quad 2\,w_{m_1} + 1\,w_{m_2},$$
$$x_{2,3}; \quad 1\,w_{m_1} + 2\,w_{m_2}.$$

This phenomenon is a type of peeling, however different from the Lorenz type bifurcation. In the Lorenz type, x_2 and x_3 peel of x_1. In the present system x_1 goes through certain bifurcation, and independent and away from x_1, the second generating singular point x_2 comes to existence in a similar fashion to the case in the system (2), Reference [6]. Global stability theorem must be satisfied for either x_1 or x_2, separately.

The stability properties of these two generating points are such that

$$x_1 \ : \ s_1 < 0, \quad s_2 < 0, \quad s_3 > 0;$$
$$2\,w_{m_1} + 1\,w_{m_2},$$
$$x_{2,3}\colon \ s_1 < 0, \quad \operatorname{Re} s_2 > 0, \quad \operatorname{Re} s_3 > 0;$$
$$1\,w_{m_1} + 2\,w_{m_2}.$$

However, separate from each other, these two points must obey the global stability theorem individually, *not cooperatively*, see [4]. This implies $1\,w_{m_2}$ of x_1 and $2\,w_{m_2}$ of $x_{2,3}$ must be balanced separately. For the point $x_{2,3}$ there are two possibilities, Fig. 3, these are the cases of a) hyperbolic neighborhood and b) elliptic neighborhood. In the case of (a) $2\,w_{m_2}$ must be absorbed by two finite points resulting in a stable limit cycle L_2. In the case (b) $1\,w_{m_2}$ is absorbed by $1\,w_{m_1}$ thus only one w_{m_2} must be absorbed by a stable exploded point E_2.

Therefore,

Conclusion 2: The generating point x_1 is surrounded by a zero f-dimensional object, a stable exploded point E_1. The second generating singular point $x_{2,3}$ is surrounded by another zero f-dimensional object, a stable exploded point E_2 (elliptic case) or a one f-dimensional object, a stable limit cycle L_2.

Thus the stable system bifurcates to yield x_1, E_1; $x_{2,3}$, E_2 (or L_2). Three systems compared exhibit

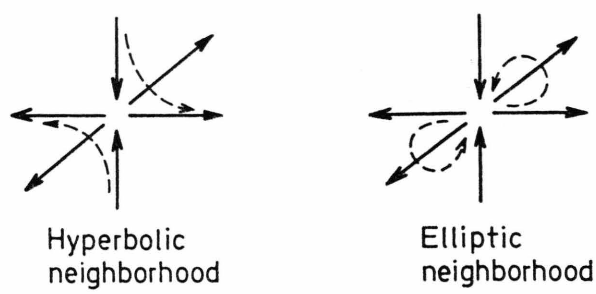

Fig. 3. Neighborhood of the saddle singular point $1\,w_{m1} \oplus 2\,w_{m2}$.

the following topological variations initially:

Lorenz (1) $x_1 \to \{x_1, x_2, x_3\}$

System (2) None $\to \{x_{2,3}\}$,

System (3) $x_1 \to \{x_1, E_1 + x_{2,3}, E_2$ (or $L_2)\}$.

The bifurcation of the second generating point of the system (3), while x_1, E_1 and E_2 (or L_2) remain unaltered, takes place when B passes the B_2* value. This is summarized below:

x_1: $s_1 < 0$, $s_2 < 0$, $s_3 > 0$,

E_1: a stable exploded point (dim $= 0$),

E_2: a stable exploded point (dim $= 0$), (or stable limit cycle (dim $= 1$)),

$x_2 \to x_2$: $s_1 < 0$, $\mathrm{Re}\, s_2 > 0$, $\mathrm{Re}\, s_3 > 0$; $1\, w_{m_1} + 2\, w_{m_2}$,

x_3: $s_1 < 0$, $\mathrm{Re}\, s_2 > 0$, $\mathrm{Re}\, s_3 > 0$; $1\, w_{m_1} + 2\, w_{m_2}$,

L_3: a stable limit cycle (dim $= 1$).

Conclusion 3: x_2 and x_3 cooperatively result in $2\, w_{m_1} + 4\, w_{m_2} \to 2\, w_{m_2}$. Thus, a limit cycle L_3 a one dimensional object surrounds x_2 and x_3.

Therefore the following three cases are possible:

1. x_1,
2. $\{x_1, E_1\} + \{x_{2,3}, E_2$ (or $L_2)\}$,
3. $\{x_1, E_1\} + \{x_2, x_3, E_2$ (or L_2), $L_3\}$.

In these cases the Poincaré conditions are satisfied, [3]. Moreover, the global stability theorem [7] is satisfied throughout the alterations. The limit cycle L_3 may be similar to the limit cycle in system (2), thus it may be a limit bundle, [6].

We should also notice that if one looks at the stability property of the three singular points without making the distinction made above, one finds the collective stability characteristics of x_1, x_2 and x_3 as

x_1: $2\, w_{m_1} + 1\, w_{m_2}$,

x_2: $1\, w_{m_1} + 2\, w_{m_2}$,

x_3: $1\, w_{m_1} + 2\, w_{m_2}$,

$4\, w_{m_1} + 5\, w_{m_2} \to 1\, w_{m_2}$.

Thus the object which we identified as divided into parts E_1, E_2 (or L_2) and L_3 are embedded into the exploded point E,

$$E = E_1 \oplus E_2 \,(\text{or } L_2) \oplus L_3\,.$$

This would imply that in order to ofset L_3 (and L_2), there must be unstable limit cycle(s) L^{u} embedded in E, however their existence can only be suggested indirectly as argued above. The limit cycle(s) as well as the exploded points are observed by simulation in [5], particularly for the special case of $d = 0$, i.e., $B = 0$, see Table 1. One of the exploded points is illustrated in Figure 4.

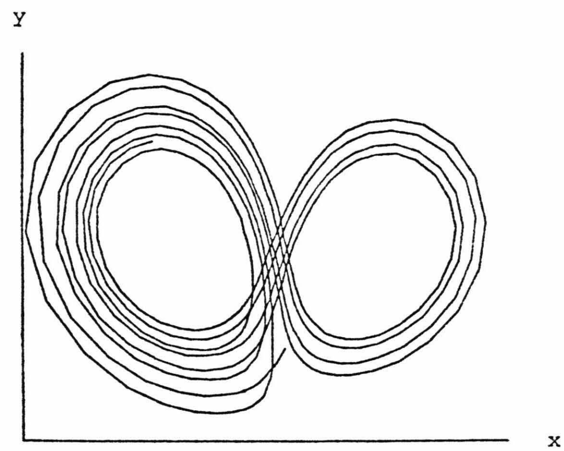

Fig. 4. An exploded point of the system (3) with the parameter values, $a = 0.1$, $b = 0.07$, $c = 0.38$, $d = 0.0$.

In summary we have shown that following the second bifurcation the three systems considered yield the cases below:

The system (1): Cooperative peeling of multiple singular points, has a unique exploded point [1, 3 and 4].

The system (2): Degenerate peeling of a single singular point, has a unique limit bundle [6].

The system (3): Independent peeling of multiple singular points, has multiple exploded points and limit bundles.

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